Ageing in disordered magnets and local scale-invariance

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Abstract. – The ageing of the bond-disordered two-dimensional Ising model quenched to below its critical point is studied through the two-time autocorrelator and thermoremanent magnetization (TRM). The corresponding ageing exponents are determined. The form of the scaling function of the TRM is well described by the theory of local scale-invariance.

A spin system quenched to below or exactly at its critical temperature T_c undergoes ageing, that is time-translation invariance is broken and observables such as correlation or response functions display dynamical scaling. While systematic studies of ageing appear to have been carried out first in studies of the mechanical response of glassy systems [1], it has been realized relatively recently that a similar phenomenology also applies for simple ferromagnets without disorder, see [2–5] for reviews. In these simpler systems, for both $T = T_c$ and $T < T_c$ there is a single time-dependent length-scale growing with time as $L(t) \sim t^{1/z}$ where z is the dynamical exponent. $L(t)$ is identified with the typical linear size of correlated or ordered clusters, respectively. It is common to study ageing through the two-time autocorrelation function $C(t,s) = \langle \phi(t,\boldsymbol{r})\phi(s,\boldsymbol{r})\rangle$ or the two-time linear autoresponse function $R(t,s) = \delta\langle\phi(t,\boldsymbol{r})\rangle/\delta h(s,\boldsymbol{r})|_{h=0}$. Here $\phi(t,\boldsymbol{r})$ denotes the space-time-dependent order-parameter, $h(s,\boldsymbol{r})$ is the space-time-dependent conjugate magnetic field, t is referred to as *observation time* and s as *waiting time*. For ageing systems, averages depend on both t and s and not merely on the difference $\tau = t-s$. In the ageing regime $t, s \gg t_{\rm micro}$ and $t-s \gg t_{\rm micro}$, where $t_{\rm micro}$ is some microscopic time scale, dynamical scaling holds (*at* equilibrium $a = b$)

$$C(t,s) = s^{-b} f_C(t/s) \ , \ R(t,s) = s^{-1-a} f_R(t/s) \tag{1}$$

such that the scaling functions $f_{C,R}(y)$ satisfy the following asymptotic behaviour

$$f_C(y) \sim y^{-\lambda_C/z} \ , \ f_R(y) \sim y^{-\lambda_R/z} \tag{2}$$

as $y \to \infty$ and where λ_C and λ_R, respectively, are known as the autocorrelation [6, 7] and autoresponse exponents [8]. For non-conserved dynamics (model A), these exponents are independent of the equilibrium exponents and of z [9]. In simple magnets without disorder and with short-ranged initial conditions one finds $\lambda_C = \lambda_R$, see [2, 10] and this has been reconfirmed in a recent second-order perturbative analysis of the time-dependent Ginzburg-Landau equation [11]. On the other hand, for either long-ranged initial correlations in ageing ferromagnets [8] or else for disordered systems such as the random-phase sine-Gordon model [12], λ_C and λ_R are known to be different from each other. Finally, if $T < T_c$, then for simple magnets with short-ranged equilibrium correlation functions (such as the Ising model in $d > 1$ dimensions) one has $b = 0$ and standard scaling arguments show that $a = 1/z$. On the other hand, if $T = T_c$, then $a = b = (d-2+\eta)/z$ where η is a standard equilibrium critical exponent.

Furthermore, according to the theory of local scale-invariance (LSI) [13] the response functions of ageing systems with an algebraic growth law $L(t) \sim t^{1/z}$ are expected to transform covariantly under local space-time transformations $t \mapsto (1+\varepsilon)^z t$, $\boldsymbol{r} \mapsto (1+\varepsilon)\boldsymbol{r}$ with an infinitesimal $\varepsilon = \varepsilon(t, \boldsymbol{r})$. Restricting the admitted transformations such that time-translations are excluded (as is natural to do in the context of ageing) leads to the following prediction of the scaling function $f_R(y)$, up to normalization [10, 13–15]

$$f_R(y) = y^{1+a'-\lambda_R/z} \, (y-1)^{-1-a'} \tag{3}$$

where a' is a new exponent. Over the past few years, evidence confirming the prediction (3) has been accumulating. First, consider phase-ordering kinetics with a non-conserved order-parameter (where $z = 2$ [16]). Then the prediction (3) with $a = a'$ was confirmed in the exactly solvable random walk and the spherical model [17] and numerically in the $2D/3D$ Ising [18] and XY [19] models and the $2D$ q-states Potts model with $q = 2, 3, 8$ [20]. Second, consider non-equilibrium critical dynamics with a non-conserved order-parameter, where in general $z \neq 2$. Again, in many spin systems eq. (3) is either exactly reproduced in solvable models or describes very well the numerical data in in a large variety of systems, in general with a and a' being distinct, including ferromagnets such as the Glauber-Ising model in $d = 1, 2, 3$ dimensions, the OJK approximation, the spherical model and glassy systems such as the Frederikson-Anderson model and the critical Ising spin glass, see the references quoted in [15]. In addition, ferromagnets in restricted gometries have also been studied [21, 22]. Finally, tests of eq. (3) for $1D$ critical systems without detailed balance (where (1) holds for connected correlators with $a \neq b$ in general, see [23–26]) were performed for the contact process (for $t/s \gtrsim 1.1$) [15, 23, 27], the non-equilibrium kinetic Ising model [26] and bosonic models [28]. We stress that both the theoretical prediction (3) of LSI as well as almost all existing tests of it([1]) assume an initial state with non-vanishing magnetization. Finally, stochastic processes such as the zero-range process [29] lack the spatial structure assumed in local scale-invariance (for instance Galilei-invariance for $z = 2$ or a convenient generalization for $z \neq 2$) and do not reproduce (3).

In this letter, we study the ageing properties of the $2D$ Ising model with bond disorder, but without frustrations, which describes the ageing of diluted magnets. The hamiltonian is

$$\mathcal{H} = -\sum_{(i,j)} J_{ij} \sigma_i \sigma_j \;\; ; \;\; \sigma_i = \pm 1 \tag{4}$$

where the random variables J_{ij} are equally distributed over the interval $[1-\varepsilon/2, 1+\varepsilon/2]$ with $0 \leq \varepsilon \leq 2$. We consider a square lattice with 300×300 spins, which is large enough

([1])In the $1D$ critical contact process a full lattice was used as initial state. It is conceivable that this distinct initial condition might cause the departure seen from eq. (3) in the region $t/s \lesssim 1.1$ in the contact process [15, 27] but the question is still open.

TABLE I – *Quantities describing the ageing behaviour of the disordered 2D Ising model.*

ε	T	b	a	λ_C/z	λ_R/z	r_0	r_1	$1/z$
0.5	1.0	0.005(2)	0.398(5)	0.570(5)	0.61(1)	0.021(1)	0	0.400
	0.8	0.001(2)	0.382(4)	0.550(5)	0.595(10)	0.020(1)	0	0.381
	0.6	-0.008(2)	0.353(4)	0.520(5)	0.58(1)	0.022(1)	0	0.353
	0.4	-0.010(2)	0.310(5)	0.460(5)	0.52(1)	0.029(2)	0.008(1)	0.308
1.0	1.0	-0.014(2)	0.330(5)	0.490(5)	0.51(1)	0.021(1)	-0.009(1)	0.333
	0.8	-0.014(2)	0.308(4)	0.450(5)	0.49(1)	0.019(1)	-0.006(1)	0.308
	0.6	-0.015(2)	0.277(6)	0.380(5)	0.46(1)	0.020(1)	-0.007(1)	0.273
	0.4	-0.013(2)	0.22(1)	0.290(5)	0.375(10)	0.026(2)	-0.014(3)	0.222
2.0	1.0	-0.021(1)	0.24(2)	0.320(5)	0.33(1)	0.048(2)	-0.048(4)	0.250
	0.8	-0.018(1)	0.22(2)	0.270(5)	0.30(1)	0.093(3)	-0.042(4)	0.222
	0.6	-0.012(1)	0.17(2)	0.220(5)	0.27(1)	0.194(4)	-0.033(3)	0.188

such that no finite-size effects are notable. It is known that this model has a second-order phase-transition and that the value of of the critical temperature $T_c = T_c(\varepsilon)$ does not change much with ε, since $\langle J_{ij} \rangle = 1$, such that $2 \leq T_c(\varepsilon) \leq 2.269$ [30]. The kinetics (non-conserved order-parameter) is described through a standard heat-bath algorithm. Throughout, we use a fully disordered initial state and average over typically 10000 different realizations of the bond disorder and of the thermal noise. In this work, we shall limit ourselves to quenches deep into the ordered phase $T < T_c(\varepsilon)$. Recently detailed studies were carried out by Paul, Puri and Rieger [30, 31] on the growth law in the low-temperature phase. Assuming that the energy barriers for domain-wall motion depend logarithmically on the linear domain-size $L = L(t)$, they proposed for a non-conserved order-parameter the growth law, valid for $t \to \infty$ [30]

$$L(t) \sim t^{1/z} \quad , \quad z = z(T, \varepsilon) = 2 + \varepsilon/T \tag{5}$$

and they confirmed this through extensive simulations of the single-time correlation functions. It will be one of our objectives to further test this relationship. In addition, we see that the value of the dynamical exponent can be easily changed so that tests of the LSI-prediction (3) for a large range of values of z becomes possible. This is particularly interesting since in previous tests of LSI in nonequilibrium critical dynamics the value of z always remained quite close to 2, the largest deviations occurring in the $1D$ contact process, where $z \simeq 1.6$.

We begin the discussion of our results by considering the two-times autocorrelation function $C(t, s)$. In figure 1a we plot our data taken for $\varepsilon = 2$ and $T = 1$ of the autocorrelator over against t/s. In constrast to what is found in phase-ordering for simple, non-disordered magnets, no data-collapse occurs here. Comparing with the anticipated long-time scaling (1), this implies that the exponent b cannot vanish, if that scaling form is valid at all. In figure 1b we show that a relatively good, but not perfect scaling can be achieved with a non-vanishing effective value of b. Still, in spite of the large waiting times used, apparently there remain important finite-time corrections to the anticipated scaling form (1). A forthcoming study [32] of the site-disordered Ising model rather suggests $C(t, s) \simeq s^{-b(t/s)} f_C(t/s)$ with two continuous functions f_C and b. Phenomenologically, this certainly achieves a better collapse than with a constant b. More detailed studies on the scaling of $C(t, s)$ will be needed.

Assuming the standard scaling form (1,2) being valid for C, we extract the value of the autocorrelation exponent λ_C/z, listed together with effective values for b in table I.

We now turn to a discussion of the scaling of the linear response. While a direct calculation

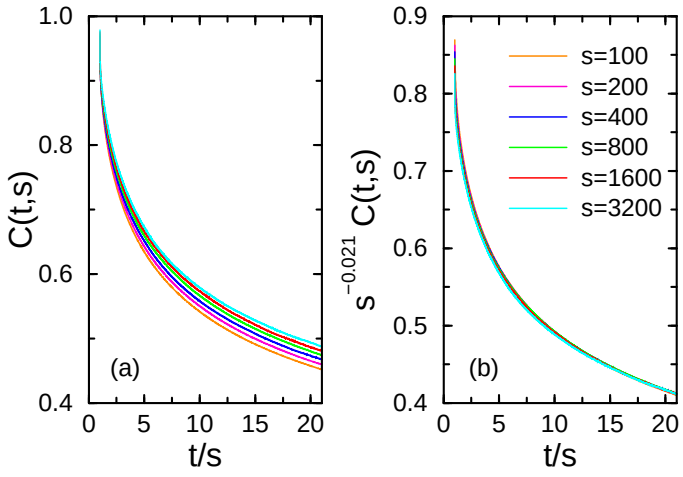

Fig. 1 – Autocorrelation for $\varepsilon = 2$ and $T = 1$ as a function of t/s for various waiting times s. For $b = 0$ no data collapse is observed (a), whereas for $b = -0.021$ the data collapse nicely for large values of t/s (b).

of the functional derivative for R would produce extremely noisy data, it is of interest to consider an integrated reponse function [33] where much of the noise is smoothed out by the integration. Here we work with the thermoremanent magnetization $M_{\rm TRM}(t,s)$ which is obtained after applying a small spatially random magnetic field with amplitude $h_0 = 0.05$. From our experience with the scaling of $M_{\rm TRM}(t,s)$ in the phase-ordering of simple magnets, we expect from (1,2) the scaling behaviour [34, 35]

$$M_{\rm TRM}(t,s) = h_0 \int_0^s {\rm d}u\, R(t,u) = r_0 s^{-a} f_M(t/s) + r_1 s^{-\lambda_R/z} g_M(t/s) \quad , \quad g_M(y) \simeq y^{-\lambda_R/z} \tag{6}$$

such that the asymptotic behaviour $f_M(y) \sim y^{-\lambda_R/z}$ holds true for $y \to \infty$. In simple magnets, the leading correction term included in (6) often is quite sizeable and must be subtracted off before a reliable estimate of the scaling function $f_M(y)$ can be obtained.

By analogy with the simple Ising model, we determined the non-universal constants r_0 and r_1 by fixing $y = t/s$. Then a fit was made from the plot of $M_{\rm TRM}(ys,s)$ over against s. The results collected in table I have been obtained by averaging over at least three different values of y. As a first example, we show in figure 2a the thermoremanent magnetization $M_{\rm TRM}(ys,s)$ for $\varepsilon = 0.5$ and $T = 1$. It turns out that in this case already the raw data scale very well which means that for the chosen values of the parameters, the finite-time corrections to the leading scaling $M_{\rm TRM}(t,s) \sim s^{-a} f_M(t/s)$ are negligible. We find the ageing exponent $a = 0.398(5)$. In phase-ordering kinetics, the dynamics of systems with short-ranged spatial correlators is determined by the motion of the domain walls only [2]. Then simple dimensional analysis yields the relation $a = 1/z$. Comparing this with the values of a for which a data collapse is achieved, we find indeed a complete agreement with

$$a = a(T,\varepsilon) = z(T,\varepsilon)^{-1} \tag{7}$$

where $z = z(T,\varepsilon)$ is given by eq. (5), see the last column in table I. We see that this agreement holds to within our numerical precision for all the values of T and ε we have considered. In

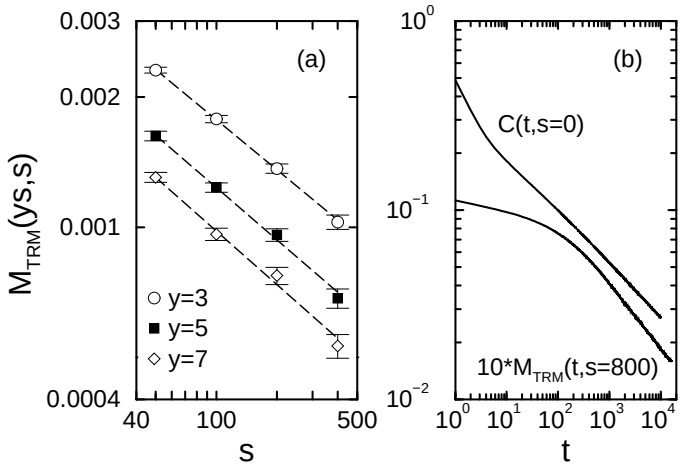

Fig. 2 – (a) Thermoremanent magnetization $M_{\rm TRM}(ys,s)$ as a function of s for $\varepsilon = 0.5$ and $T = 1.0$ and different values of y. The slopes of the fitted straight lines yield the average value 0.398(5) for the exponent a. (b) Time-dependent behaviour of the autocorrelator $C(t, s = 0)$ and of the thermoremanent magnetization $M_{\rm TRM}(t, s = 800)$ for $\varepsilon = 1$ and $T = 0.4$. The slopes of the straight parts are different for large values of t, indicating that the exponents λ_R/z and λ_C/z are different. For illustrative reasons M_{TRM} has been multiplied by 10.

this way, we confirm the earlier conclusion of Rieger *et al.* [30,31] on the scaling of the domain size $L(t)$. We also recall that for the simple Ising model, one has $a = 1/z = 1/2$.([2])

We point out that in contrast to the autocorrelation function, the integrated response displays a completely conventional dynamical scaling according to eqs. (1,2).

Having understood how to achieve dynamical scaling for the linear response, we next look at the asymptotic behaviour $f_M(y) \sim y^{-\lambda_R/z}$ for $y \to \infty$. We collect the results for the autoresponse exponent λ_R/z in table I and now compare with the values of λ_C/z we have determined before. It can be clearly seen that the autocorrelation and autoreponse exponents are different for the model at hand. We further illustrate this in figure 2b where the autocorrelation function $C(t, 0) \sim t^{-\lambda_C/z}$ for t large enough and the thermoremanent magnetization $M_{\rm TRM}(t, 800) \sim t^{-\lambda_R/z}$ are compared for $\varepsilon = 1$ and $T = 0.4$ in a log-log plot. Clearly the slopes for t large and hence the two exponents are different. This behaviour sets the kinetics of the bond-disordered Ising model apart from what is known for the phase-ordering of all simple magnets with short-ranged initial correlations. We did not anticipate this finding but recall that the random-phase sine-Gordon model also shows $\lambda_C \neq \lambda_R$ slightly below its critical temperature [12].

After these preparations, we are ready to study in more detail the form of the scaling function $f_M(y)$ of the integrated linear response. It is straightforward to integrate the prediction (3) of local scale-invariance with $a = a'$. We have ($_2F_1$ is a hypergeometric function)

$$f_M(y) = y^{-\lambda_R/z}\, {}_2F_1\left(1 + a, \frac{\lambda_R}{z} - a; \frac{\lambda_R}{z} - a + 1; \frac{1}{y}\right) \tag{8}$$

([2])It was claimed that because of dangerous irrelevant variables, the above-mentioned simple picture might not be valid in phase-ordering, leading to $a \neq 1/z$. For example in the $2D$ Ising model $a = 1/4$ was proposed [36]. Detailed studies of the simple $2D$ Ising model reconfirmed $a = 1/z = 1/2$ [18,20,34,37,38], in agreement with the standard picture and *contra* the claim of [36]. The support of eq. (7) in the diluted Ising model provides further evidence against the claim raised in [36].

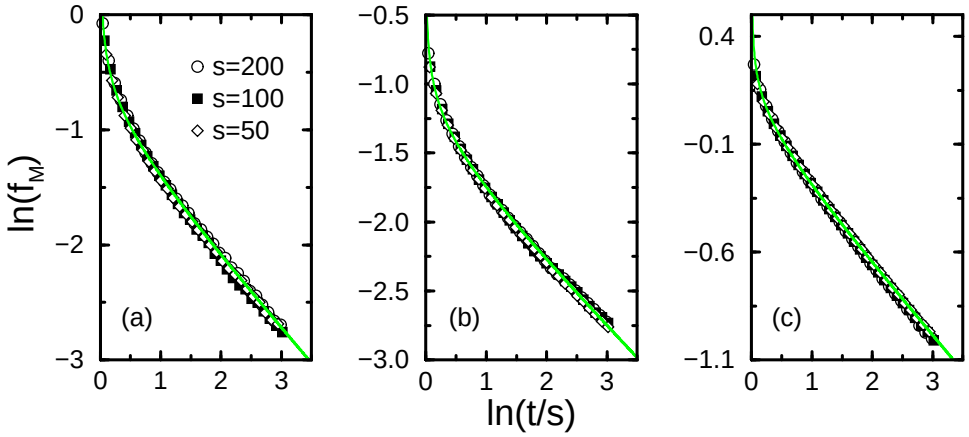

Fig. 3 – Comparison of the numerically determined scaling functions $f_M(t/s)$ with the predictions coming from local scale-invariance (full lines). (a) $\varepsilon = 0.5$ and $T = 1$, (b) $\varepsilon = 1$ and $T = 0.6$, (c) $\varepsilon = 2$ and $T = 1$. In the first case corrections to scaling are negligibly small. Error bars are smaller than the sizes of the symbols.

In this expression, the values of the exponents and also the normalization are already fixed, see table I. We can now compare the data for the scaling function with the prediction (8) of LSI. In figure 3 examples for this comparison are shown for three choices of ε and T. In the first case we have already seen that finite-time corrections to scaling were negligibly small and we now see from figure 3 that the form of the scaling function is perfectly described by LSI. In the two other cases, the finite-time corrections are notable and actually quite sizeable for $\varepsilon = 2$. Subtracting them, a very good scaling behaviour is found. Again, the curve (8) matches the data very well. This agreement with LSI could not hold if (5) or (7) were invalid.

It is remarkable, in view of the very large values of z, how well the theory of local scale-invariance describes the data. From a field-theoretical perspective, it was advanced [4] that a symmetry principle such as local scale-invariance could at best hold at the mean-field level, where $z = 2$. The present example, using a non-integrable spin system with values of the dynamical exponent which can become very large, rather suggests that the idea of extending dynamical scaling to a more local form should indeed be capable of faithfully reproducing at least the linear responses of physically quite distinct systems. At present, it appears that the main requirements on the models to be studied are an algebraic growth law $L(t) \sim t^{1/z}$ of the linear size of the correlated clusters and a vanishing initial magnetization. The present study closes a gap between the simple, non-disordered magnets and on the other side *critical* spin glasses [14] where local scale-invariance has been successfully tested before.

In summary, we have studied the ageing behaviour of a bond-disordered two-dimensional Ising model quenched to temperatures far below its critical temperature $T_c(\varepsilon)$. While, the linear response as studied by the thermoremanent magnetization was seen to be completely compatible with standard dynamical scaling, large finite-time corrections to standard dynamical scaling were seen in the two-time autocorrelation function. Our data appear to be compatible with the standard relation $a = 1/z$ of the ageing exponent a with the dynamical exponent $z(T, \varepsilon)$ given by eq. (5). We also encountered two surprises: first, assuming (1), the effective ageing exponent b was found to be non-vanishing (and negative) and second, the autocorrelation and autoresponse exponents are different, $\lambda_C \neq \lambda_R$. In spite of the large

range of values of z, the form of the scaling function of the thermoremanent magnetization is described to within the numerical accuracy by the theory of local scale-invariance.

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